

Explicit Model Construction for Saturated Constrained Horn Clauses

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Abstract

Clause sets saturated by hierarchic superposition do not offer an explicit model representation, rather the guarantee that all non-redundant inferences have been performed without deriving a contradiction. We present an approach to explicit model construction for saturated constrained Horn clauses. Constraints are in linear arithmetic, the first-order part is restricted to a function-free language. The model construction is effective and clauses can be evaluated with respect to the model. Furthermore, we prove that our model construction produces the least model.

1 Introduction

Constrained Horn Clauses (CHCs) combine logical formulas with constraints over various domains, e.g. linear real arithmetic, linear integer arithmetic, equalities of uninterpreted functions [13]. This formalism has gained widespread attention in recent years due to its applications in a variety of fields, including program analysis and verification: safety, liveness, and termination [41, 16], complexity and resource analysis [37], intermediate representation [22], and software testing [39]. Technical controls, so called *Supervisors*, like an engine electronic control unit, or a lane change assistant in a car [8, 7] can be modelled, run, and proven safe. Thus, CHCs are a powerful tool for reasoning about complex systems that involve logical constraints, and they have been used to solve a wide range of problems.

A failed proof attempt of some conjecture or undesired run points either to a bug in the model, conjecture or in the modelled system. In this case investigation of the cause of the unexpected result or behavior is crucial. Building an explicit model of the situation that can then be effectively queried is an important means towards a repair. This is exactly the contribution of this paper: We show how

to build explicit models out of saturated CHC clause sets that can then be effectively queried, i.e., clauses can be effectively evaluated with respect to the model.

Reasoning in CHCs, or first-order logic fragments extended with theories, has a long tradition. There are approaches to clausal, resolution style reasoning such as superposition [2, 23, 31], sequent style reasoning [40], and reasoning based on explicit model assumptions [4, 9]. More recently, specific approaches to CHCs have been developed that can also consider inductive aspects. They fall into one of two categories [13, 6]: bottom-up procedures [24] or top-down procedures [26, 38, 30]. *Bottom-up procedures* are based on forward inferences. For instance, hierarchic unit resolution to CHCs would constitute a bottom-up procedure for CHCs. Bottom-up procedures return “unsatisfiable” as soon as they derive a clause of the form $\Lambda \parallel \perp$, where Λ is a satisfiable constraint, the first-order part \perp of the clause is unsatisfiable, and they return “satisfiable” if no such clause was derived and no more inferences are possible, i.e., the clause set is saturated. Many bottom-up procedures can construct the least model based on a saturated clause set this way [13]. For instance, the constraint unit clauses that are the result of saturating with hierarchic unit resolution describe together the least model. However, for many interesting cases, bottom up saturation does not terminate and the saturated clause sets are infinite. *Top-down procedures* are based on backward inferences, i.e., they start their inferences from so-called *goal clauses* (typically purely negative clauses) and continuously infer new goal clauses. For instance, an extension of SLD resolution [32, 35, 29] to CHCs would constitute a top-down procedure for CHCs. Chains of backward inferences can then be pruned early with the help of cyclic induction or interpolants [6]. Moreover, over-approximations of the least model can be extracted from such a pruning and tested for satisfiability. Some contemporary methods for solving CHCs combine top-down and bottom-up approaches, e.g., via *abstraction refinement* [13]. CHCs can be extended to a full fledged programming language as done in constraint logic programming [28, 13].

In contrast to most CHC reasoning procedures, superposition is neither a bottom-up nor a top-down procedure. Instead of restricting itself to forward/backward inferences, superposition restricts its inferences based on an ordering on the ground literals. The finite saturation concept of superposition is powerful: there are saturated CHCs with linear arithmetic constraints where it is undecidable whether a simple ground fact is a consequence of the saturated set [14, 27]. Superposition is a decision procedure for various first-order logic fragments, e.g., [2, 23, 25], even if they are extended with theories [34, 20, 9]. The decision results for SCL (Simple Clause Learning) reasoning [9] are also based on the completeness of hierarchic superposition. The proof for refutational completeness of hierarchic superposition even implies the existence of a model if the clause set is saturated and does not contain the empty clause [33, 5, 3]. However, the definition of the model from the completeness proof is based on an infinite axiomatization of the background theory. It is therefore not suited for an effective, explicit model representation.

In this work, we present an automated model building approach that yields an explicit and finite model representation for finitely saturated Horn clause sets of linear arithmetic combined with the Bernays-Schönfinkel fragment. This fragment is equivalent to CHCs with linear arithmetic constraints. Recall that although satisfiability in this fragment is undecidable [14, 27], in general, for a

finitely saturated set we can effectively construct such a representation. This enables effective evaluation of clauses with respect to the model and therefore supports explanation and bug finding in case of failed refutations. The approach we present does not exploit features of linear arithmetic beyond equality and the existence of a well-founded order for the theories' universe. The results may therefore be adapted to other constraint domains.

Recursive predicate definitions cannot be finitely saturated, in general. This is a result of the expressiveness of the language: a single monadic predicate together with constraints of the form $y = cx$ or $y > c$ for some number c and recursive clauses, already yields undecidability of CHCs modulo linear arithmetic [27]. As a result, it is possible to construct for any CHC reasoning approach small recursive examples where also our approach does not yield termination. Still, our model construction approach and CHC reasoning in general are useful in practice. There are many real world applications that are either non-recursive or only depend on recursion that reaches a fixed point after a finite number of recursive steps [9, 8, 7]. Naturally, such applications make saturation possible and therefore fall into the scope of our method. Moreover, our method is also of theoretical interest because it is the first explicit model construction approach for superposition that is based on saturation, goes beyond ground clauses, and includes theory constraints. In the future, we plan to use this approach as the basis for a more general model construction approach that also works on more expressive fragments of first-order logic modulo theories.

There is a tradition in automated model building in first-order logic [19, 11]. A model should in particular enable effective evaluation of clauses [11]. This is fulfilled by our finite model representation to CHCs. In detail, our contributions are: (i) We propose an approach to automatically build models for clause sets that are saturated under hierarchical superposition up to redundancy, (ii) the resulting models are effective, in the sense that they allow clause evaluation, and (iii) we show that our approach produces the least model.

The paper is organized as follows. In Section 2 we clarify notation and preliminaries. The main contribution is presented in Section 3. We conclude in Section 4.

2 Preliminaries and Notation

We briefly recall the basic logical formalisms and notations we build upon [8]. Our starting point is a standard first-order language with *variables* (denoted x, y, z), *predicates* (denoted P, Q) of some fixed *arity*, and *terms* (denoted t, s). An *atom* (denoted A) is an expression $P(t_1, \dots, t_n)$ for a predicate P of arity n . When the terms t_1, \dots, t_n in $P(t_1, \dots, t_n)$ are not relevant in some context, we also write $P(*)$. A *positive literal* is an atom A and a *negative literal* is a negated atom $\neg A$. We define $\text{comp}(A) = \neg A$, $\text{comp}(\neg A) = A$, $|A| = A$ and $|\neg A| = A$. Literals are usually denoted L, K . We sometimes write literals as $[\neg]P(*)$, meaning that the sign of the literal is arbitrary, often followed by a case distinction. Formulas are defined in the usual way using quantifiers \forall, \exists and the boolean connectives $\neg, \vee, \wedge, \rightarrow$, and \equiv . The logic we consider does not feature an equality predicate.

A *clause* (denoted C, D) is a universally closed disjunction of literals $A_1 \vee \dots \vee A_n \vee \neg B_1 \vee \dots \vee \neg B_m$. A clause is *Horn* if it contains at most one positive

literal, i.e. $n \leq 1$. In Section 3, all clauses considered are Horn clauses. If Y is a term, formula, or a set thereof, $\text{vars}(Y)$ denotes the set of all variables in Y , and Y is *ground* if $\text{vars}(Y) = \emptyset$. Analogously $\Pi(Y)$ is the set of predicate symbols occurring in Y .

The *Bernays-Schönfinkel Clause Fragment* (BS) in first-order logic consists of first-order clauses where all terms are either variables or constants. The *Horn Bernays-Schönfinkel Clause Fragment* (HBS) is further restricted to Horn clauses.

A *substitution* σ is a function from variables to terms with a finite domain $\text{dom}(\sigma) = \{x \mid x\sigma \neq x\}$ and codomain $\text{codom}(\sigma) = \{x\sigma \mid x \in \text{dom}(\sigma)\}$. We denote substitutions by σ, τ . The application of substitutions is often written postfix, as in $x\sigma$, and is homomorphically extended to terms, atoms, literals, clauses, and quantifier-free formulas. A substitution σ is *ground* if $\text{codom}(\sigma)$ is ground. Let Y denote some term, literal, clause, or clause set. A substitution σ is a *grounding* for Y if $Y\sigma$ is ground, and $Y\sigma$ is a *ground instance* of Y in this case. We denote by $\text{gnd}(Y)$ the set of all ground instances of Y . The *most general unifier* $\text{mgu}(Z_1, Z_2)$ of two terms/atoms/literals Z_1 and Z_2 is defined as usual, and we assume that it does not introduce fresh variables and is idempotent.

2.1 Horn Bernays-Schönfinkel with Linear Arithmetic

The class HBS(LRA) is the extension of the Horn Bernays-Schönfinkel fragment with linear real arithmetic (LRA). Analogously, the classes HBS(LQA) and HBS(LIA) are the extensions of the Horn Bernays-Schönfinkel fragment with linear rational arithmetic (LQA) and linear integer arithmetic (LIA), respectively. The only difference between the three classes are the sort LA their variables and terms range over and the universe \mathcal{U} over which their interpretations range. As the names already imply $\text{LA} = \text{LRA}$ and $\mathcal{U} = \mathbb{R}$ for HBS(LRA), $\text{LA} = \text{LQA}$ and $\mathcal{U} = \mathbb{Q}$ for HBS(LQA), and $\text{LA} = \text{LIA}$ and $\mathcal{U} = \mathbb{Z}$ for HBS(LIA). The results presented in this paper hold for all three classes and by HBS(LA) we denote that we are talking about an arbitrary one of them.

Linear arithmetic terms are constructed from a set \mathcal{X} of *variables*, the set of constants $c \in \mathbb{Q}$ (if in HBS(LRA) or HBS(LQA)) or $c \in \mathbb{Z}$ (if in HBS(LIA)), and binary function symbols $+$ and $-$ (written infix). Additionally, we allow multiplication \cdot if one of the factors is a constant. Multiplication only serves us as syntactic sugar to abbreviate other arithmetic terms, e.g., $x+x+x$ is abbreviated to $3 \cdot x$. Atoms in HBS(LA) are either *first-order atoms* (e.g., $P(13, x)$) or (*linear*) *arithmetic atoms* (e.g., $x < 42$). Arithmetic atoms are denoted by λ and may use the predicates $\leq, <, \neq, =, >, \geq$, which are written infix and have the expected fixed interpretation. *First-order literals* and related notation is defined as before. *Arithmetic literals* coincide with arithmetic atoms, since the arithmetic predicates are closed under negation, e.g., $\neg(x \geq 42) \equiv x < 42$.

HBS(LA) clauses are defined as for HBS but using HBS(LA) atoms. We often write clauses in the form $\Lambda \parallel C$ where C is a clause solely built of free first-order literals and Λ is a multiset of LA atoms called the *constraint* of the clause. A clause of the form $\Lambda \parallel C$ is therefore also called a *constrained clause*.

The fragment we consider in Section 3 is restricted even further to *abstracted* clauses: For any clause $\Lambda \parallel C$, all terms in C must be variables. Put differently, we disallow any arithmetic function symbols, including numerical constants, in C . Abstraction, e.g. rewriting $x \geq 3 \parallel P(x, 1)$ to $x \geq 3, y = 1 \parallel P(x, y)$, is al-

ways possible. This is not a theoretical limitation, but allows us to formulate our model construction operator in a more concise way. We assume abstracted clauses for theory development, but we prefer non-abstracted clauses in examples for readability, e.g., a unit clause $P(3, 5)$ is considered in the development of the theory as the clause $x = 3, y = 5 \parallel P(x, y)$.

In contrast to other works, e.g. [10], we do not permit first-order constants, and consequently also no variables that range over the induced herbrand universe. All variables are arithmetic in the sense that they are interpreted by \mathcal{U} . In the absence of equality, it is possible to simulate variables over first-order constants, by e.g. numbering them, i.e. defining a bijection between \mathbb{N} and constant symbols.

The semantics of $\Lambda \parallel C$ is as follows:

$$\Lambda \parallel C \quad \text{iff} \quad \left(\bigwedge_{\lambda \in \Lambda} \lambda \right) \rightarrow C \quad \text{iff} \quad \left(\bigvee_{\lambda \in \Lambda} \neg \lambda \right) \vee C$$

For example, the clause $x > 1 \vee y \neq 5 \vee \neg Q(x) \vee R(x, y)$ is also written $x \leq 1, y = 5 \parallel \neg Q(x) \vee R(x, y)$. The negation $\neg(\Lambda \parallel C)$ of a constrained clause $\Lambda \parallel C$ where $C = A_1 \vee \dots \vee A_n \vee \neg B_1 \vee \dots \vee \neg B_m$ is thus equivalent to $(\bigwedge_{\lambda \in \Lambda} \lambda) \wedge \neg A_1 \wedge \dots \wedge \neg A_n \wedge B_1 \wedge \dots \wedge B_m$. Note that since the neutral element of conjunction is \top , an empty constraint is thus valid, i.e. equivalent to true. In analogy to the empty clause in settings without constraints, we write \square to mean any and all clauses $\Lambda \parallel \perp$ where Λ is satisfiable, which are all unsatisfiable.

An *assignment* for a constraint Λ is a substitution (denoted β) that maps all variables in $\text{vars}(\Lambda)$ to numbers $c \in \mathcal{U}$. An assignment is a *solution* for a constraint Λ if all atoms $\lambda \in (\Lambda\beta)$ evaluate to true. A constraint Λ is *satisfiable* if there exists a solution for Λ . Otherwise it is *unsatisfiable*.

We assume *pure* input clause sets, which means the only constants of our sort LA are concrete rational numbers. Irrational numbers are not allowed by the standard definition of the theory. Fractions are not allowed if $\text{LA} = \text{LIA}$. Satisfiability of pure HBS(LA) clause sets is semi-decidable, e.g., using *hierarchical superposition* [3] or *SCL(T)* [9]. Impure HBS(LA) is no longer compact and satisfiability becomes undecidable, but its restriction to ground clause sets is decidable [21]. Note that pure HBS(LA) clauses correspond to *constrained Horn clauses (CHCs)* with LA as background theory.

All arithmetic predicates and functions are interpreted in the usual way. An interpretation of HBS(LA) coincides with \mathcal{A}^{LA} on arithmetic predicates and functions, and freely interprets free predicates. For pure clause sets this is well-defined [3]. Logical satisfaction and entailment is defined as usual, and uses similar notation as for HBS.

Example 1. *The clause $y \geq 5, x' = x + 1 \parallel S_0(x, y) \rightarrow S_1(x', 0)$ is part of a timed automaton with two clocks x and y modeled in HBS(LA). It represents a transition from state S_0 to state S_1 that can be traversed only if clock y is at least 5 and that resets y to 0 and increases x by 1.*

2.2 Ordering Literals and Clauses

In order to define redundancy for constrained clauses, we need an *order*: Let \prec_{Π} be a total, well-founded, strict ordering on predicate symbols and let $\prec_{\mathcal{U}}$ be a total, well-founded, strict ordering on the universe \mathcal{U} . (Note that \prec cannot

be the standard ordering $<$ because it is not well-founded for \mathbb{Z} , \mathbb{Q} , or \mathbb{R} . In the case of \mathbb{R} , the existence of such an order is even dependent on whether we assume the axiom of choice [17].) We extend these orders step by step. First, to atoms, i.e., $P(\vec{a}) \prec Q(\vec{b})$ if $P \prec_{\Pi} Q$ or $P = Q$, $\vec{a}, \vec{b} \in \mathcal{U}^{|\vec{a}|}$, and $\vec{a} \prec_{\text{lex}} \vec{b}$, where \prec_{lex} is the lexicographic extension of $\prec_{\mathcal{U}}$. Next, to literals with a strict precedence on the predicate and the polarity, i.e.,

$$P(\vec{t}) \prec \neg P(\vec{s}) \prec Q(\vec{u}) \quad \text{if } P \prec Q$$

independent of the arguments of the literals. Then, take the multiset extension to order clauses. To handle constrained clauses extend the relation such that constraint literals (in our case arithmetic literals) are always smaller than first-order literals. We conflate the notation of all extensions into the symbol \prec and define \preceq as the reflexive closure of \prec . Note that \prec is only total for ground atoms/literals/clauses. However, this is sufficient for a hierarchic-superposition order [5].

Definition 1 (\prec -maximal Literal). *A literal L is called \prec -maximal in a clause C if there exists a grounding substitution σ for C , such that there is no different $L' \in C$ for which $L\sigma \prec L'\sigma$. The literal L is called strictly \prec -maximal if there is no different $L' \in C$ for which $L\sigma \preceq L'\sigma$.*

Proposition 1. *If \prec is a predicate-based ordering and C is a Horn clause, and C has a positive literal L , and L is \prec -maximal in C , then L is strictly \prec -maximal in C .*

Definition 2 (\prec -maximal Predicate in Clause). *A predicate symbol P is called (strictly) \prec -maximal in a clause C if there is a literal $[\neg]P(*) \in C$ that is (strictly) \prec -maximal in C .*

Definition 3. *Let N be a set of clauses and \prec a clause ordering. Then $N^{<C} := \{C' \in N \mid C' \prec C\}$.*

Definition 4. *Let N be a set of clauses, \prec a clause ordering, and P a predicate symbol. Then $N^{\preceq P} := \{C \in N \mid C = [\neg]Q(*) \vee C', C' \preceq [\neg]Q(*), Q \preceq P\}$.*

2.3 Hierarchical Superposition, Redundancy and Saturation

For pure HBS(LA) most rules of the (hierarchic) superposition calculus become obsolete or can be simplified. In fact, in the HBS(LA) case (hierarchic) superposition boils down to (hierarchic) ordered resolution. For a full definition of (hierarchic) superposition calculus in the context of linear arithmetic, consider SUP(LA)[1]. Here, we will only define its simplified version in the form of the hierarchic resolution rule.

Definition 5 (Hierarchical \prec -Resolution). *Let \prec be an order on literals and $\Lambda_1 \parallel L_1 \vee C_1$, $\Lambda_2 \parallel L_2 \vee C_2$ be constrained clauses. The inference rule of hierarchical \prec -resolution is:*

$$\frac{\Lambda_1 \parallel L_1 \vee C_1 \quad \Lambda_2 \parallel L_2 \vee C_2 \quad \sigma = \text{mgu}(L_1, \text{comp}(L_2))}{(\Lambda_1, \Lambda_2 \parallel C_1 \vee C_2)\sigma}$$

where L_1 is \prec -maximal in C_1 and L_2 is \prec -maximal in C_2 .

Note that in the resolution rule we do not enforce explicitly that the positive literal is strictly maximal. This is possible because in the Horn case any positive literal is strictly maximal if it is maximal in the clause.

For saturation, we need a termination condition that defines when the calculus under consideration cannot make any further progress. In the case of superposition, this notion is that any new inferences are *redundant*.

Definition 6 (Ground Clause Redundancy). *A ground clause $\Lambda \parallel C$ is redundant with respect to a set N of ground clauses and an order \prec if $N \prec \Lambda \parallel C \models \Lambda \parallel C$.*

Definition 7 (Non-Ground Clause Redundancy). *A clause $\Lambda \parallel C$ is redundant with respect to a clause set N and order \prec if for all $\Lambda' \parallel C' \in \text{gnd}(\Lambda \parallel C)$ the clause $\Lambda' \parallel C'$ is redundant with respect to $\text{gnd}(N)$.*

If a clause $\Lambda \parallel C$ is redundant with respect to a clause set N , then it can be removed from N without changing its semantics. Determining clause redundancy is an undecidable problem [9, 43]. However, there are special cases of redundant clauses that can be easily checked, e.g., tautologies and subsumed clauses.

Redundancy also means that $\mathcal{I} \models N \prec \Lambda \parallel C$ implies $\mathcal{I} \models \Lambda \parallel C$. We will exploit this fact in the model construction.

Definition 8 (Saturation). *A set of clauses N is saturated up to redundancy with respect to some set of inference rules, if application of any rules to clauses in N yields a clause that is redundant with respect to N or is contained in N .*

2.4 Interpretations

In our context, models are interpretations that satisfy (sets of) clauses. The standard notion of an interpretation is fairly opaque and interprets a predicate P as the potentially infinite set of ground arguments that satisfy P .

Definition 9 (Interpretation). *Let P be a predicate symbol of arity n . Then, $P^{\mathcal{I}}$ denotes the subset of \mathcal{U}^n for which the interpretation \mathcal{I} maps the predicate symbol P to true.*

Since our model construction approach manipulates interpretations directly, we need a notion of interpretations that always has a finite and explicit representation and for which it is possible to decide (in finite time) whether a clause is satisfied by the interpretation. Therefore, we rely on the notion of symbolic interpretations which is closely related to *\mathcal{A} -definable models* [6, Definition 7] and *constrained atomic representations* [11, Definition 5.1, pp. 236-237].

Definition 10 (Symbolic Interpretation). *Let x_1, x_2, \dots be an infinite sequence of distinct variables, i.e. $x_i \neq x_j$ for all $1 \leq i < j$. (We assume the same sequence for all symbolic interpretations.) A symbolic interpretation \mathcal{S} is a function that associates all predicate symbols P of arity n with a formula $P^{\mathcal{S}}(\vec{x})$, constructed using the usual boolean connectives over LA atoms, where the only free variables appear in $\vec{x} = (x_1, \dots, x_n)$.*

The interpretation $\mathcal{I}_{\mathcal{S}}$ corresponding to \mathcal{S} is defined by $P^{\mathcal{I}_{\mathcal{S}}} = \{(\vec{x})\beta \mid \beta \models P^{\mathcal{S}}(\vec{x})\}$ and maps the predicate symbol P to true for the subset of \mathcal{U}^n which

corresponds to the solutions of $P^S(\vec{x})$. Note that there are some interpretations for which no corresponding symbolic interpretation exists, for instance the set of prime numbers is a satisfying interpretation for $y = 2 \parallel P(y)$, but not expressible as a symbolic interpretation (in LA). As we will later see, at least any saturated set of HBS(LA) clauses either is unsatisfiable or has a symbolic interpretation that satisfies it (Section 3).

The *top interpretation*, denoted \mathcal{I}_\top , is defined as $P^{\mathcal{I}_\top} := \mathcal{U}^n$ for all predicate symbols P of arity n and corresponds to the *top symbolic interpretation*, denoted \mathcal{S}_\top , defined as $P^{\mathcal{S}_\top} := \top$ for all predicate symbols P . The *bottom interpretation* (or *empty interpretation*), denoted \mathcal{I}_\perp , is defined analogously. The interpretation of P under $\mathcal{I} \cup \mathcal{J}$ is defined as $P^{\mathcal{I} \cup \mathcal{J}} := P^{\mathcal{I}} \cup P^{\mathcal{J}}$ and corresponds to $\mathcal{I} \cup \mathcal{J}$ defined as $P^{\mathcal{S} \cup \mathcal{R}}(\vec{x}) := P^{\mathcal{S}}(\vec{x}) \vee P^{\mathcal{R}}(\vec{x})$. We write $\mathcal{I} \subseteq \mathcal{J}$ or \mathcal{I} is *included in* \mathcal{J} (resp. $\mathcal{I} \subset \mathcal{J}$ or \mathcal{I} is *strictly included in* \mathcal{J}) if $P^{\mathcal{I}} \subseteq P^{\mathcal{J}}$ (resp. $P^{\mathcal{I}} \subset P^{\mathcal{J}}$) for all predicate symbols P .

Definition 11 (Entailment of Ground Literal). *Let \mathcal{I} be an interpretation. Given a ground literal $P(a_1, \dots, a_n)$, where $a_i \in \mathcal{U}$, we write $\mathcal{I} \models P(a_1, \dots, a_n)$ if $(a_1, \dots, a_n) \in P^{\mathcal{I}}$. Conversely, we write $\mathcal{I} \not\models P(a_1, \dots, a_n)$ if $(a_1, \dots, a_n) \notin P^{\mathcal{I}}$.*

Definition 12 (Entailment of Non-Ground Literal). *Let \mathcal{I} be an interpretation. We write $\mathcal{I} \models L$, if for all grounding substitutions σ for L , we have $\mathcal{I} \models L\sigma$. Conversely, we write $\mathcal{I} \not\models L$, if there exists a grounding substitution σ for L , such that $\mathcal{I} \not\models L\sigma$.*

We overload \models for symbolic interpretations, i.e. we write $\mathcal{S} \models L$ and mean $\mathcal{I}_{\mathcal{S}} \models L$. The following function encodes a clause as an LA formula for evaluation under a given symbolic interpretation.

Definition 13 (Clause Evaluation Function). *Given a constrained clause $\Lambda \parallel C$ where $C = L_1 \vee \dots \vee L_m$, $L_i = [\neg]P_i(y_{i,1}, \dots, y_{i,n_i})$, and a symbolic interpretation \mathcal{S} , let the free variables of $P_i^{\mathcal{S}}$ be \vec{x}_i and define the following (for $1 \leq i \leq m$):*

$$\begin{aligned} \phi_i &:= \begin{cases} P_i^{\mathcal{S}} & L_i \text{ is positive} \\ \neg P_i^{\mathcal{S}} & L_i \text{ is negative (otherwise)} \end{cases} \\ \sigma_i &:= \{x_{i,j} \mapsto y_{i,j} \mid 1 \leq j \leq n_i\} \\ (\Lambda \parallel C)^{\mathcal{S}} &:= \left(\bigwedge_{\lambda \in \Lambda} \lambda \right) \rightarrow \left(\bigvee_{i=1}^m \phi_i \sigma_i \right) \end{aligned}$$

Solving $(\Lambda \parallel C)^{\mathcal{S}}$ is, essentially, the same as evaluating $\mathcal{S} \models \Lambda \parallel C$.

Proposition 2. *Given a constrained clause $\Lambda \parallel C$ with grounding β , we have*

$$\models (\Lambda \parallel C)^{\mathcal{S}} \beta \quad \text{if and only if} \quad \mathcal{S} \models (\Lambda \parallel C) \beta$$

We require two functions that manipulate LA-formulas directly to express our model construction (cf. Definition 16), i.e. to map solutions for a clause to a predicate.

Definition 14 (Projection). *Let V be a set of variables and ϕ be an LA-formula. The projection function π is defined as follows:*

$$\pi(V, \phi) := \exists x_1 \dots \exists x_n. \phi \quad \text{where } \{x_1, \dots, x_n\} = \text{vars}(\phi) \setminus V$$

This function is used when mapping the solution of a predicate w.r.t. non-maximal literals to the interpretation of the maximal literal (see Definition 16). Such projections can be solved for LRA and LQA with Loos-Weispfenning-elimination and for LIA with Cooper elimination [36, 12].

The following function helps capturing literals of the form $P(x, x)$, i.e. where one variable is shared among two arguments.

Definition 15 (Sharing). *Let (y_1, \dots, y_n) and (x_1, \dots, x_n) be tuples of variables with the same length. The function Υ , which encodes variable sharing across different arguments, is defined as follows:*

$$\Upsilon((y_1, \dots, y_n), (x_1, \dots, x_n)) := \bigwedge_{1 \leq i < j \leq n, y_i = y_j} x_i = x_j$$

Note that the equality between y_i and y_j is syntactic on variable symbols, while the equality between x_i and x_j is from LA.

2.5 Consequence and Least Model

The notion of a *least model* is common in logic programming. Horn logic programs admit a least model, which is the intersection of all models of the program (see [35, § 6, p. 36]). In our context, the least model of a set of clauses N is the intersection of all models of N . An alternative characterization of the least model of N through the least fixed point of the one-step consequence operator, which we define in analogy to $T_P^{\mathcal{D}}$ [28, Section 4] where \mathcal{D} refers to LA, see [28, Example 2.1]. For the definition of T_P concerning clauses without constraints refer to [35, § 6, p. 37] and [15]. The one-step consequence operator T_N which takes set of clauses N and an interpretation \mathcal{I} and returns an interpretation:

$$P^{T_N(\mathcal{I})} := \left\{ (\vec{y})\beta \mid \begin{array}{l} \Lambda \parallel \neg P_1(\vec{y}_1) \vee \dots \vee \neg P_n(\vec{y}_n) \vee P(\vec{y}) \in N, \\ \models \Lambda\beta, \text{ and } \mathcal{I} \models P_i(\vec{y}_i)\beta \text{ for } 1 \leq i \leq n \end{array} \right\}$$

The least fixed point of this operator exists by Tarski's Fixed Point Theorem [42]: Interpretations form a complete lattice under inclusion (supremum given by union, infimum given by intersection), and T_N is monotone.

3 Model Construction

In this section we address construction of models for HBS(LA). Throughout this section, we consider a set of constrained Horn clauses N and an order \prec as given. Our aim is to define an interpretation \mathcal{I}_N , such that

$$\mathcal{I}_N \models N \quad \text{if } N \text{ is saturated and } \square \notin N$$

Towards that goal, we define the operator $\delta(\mathcal{S}, \Lambda \parallel C' \vee P(\vec{y}))$. It takes a symbolic interpretation \mathcal{S} , and a horn clause with maximal literal $P(\vec{y})$. It results in a symbolic interpretation that accounts for $\Lambda \parallel C' \vee P(\vec{y})$.

Definition 16 (δ). *Given $\Lambda \parallel C$ where $C = C' \vee P(\vec{y})$, $P(\vec{y}) \succ C'$, and a symbolic interpretation \mathcal{S} , let $n := |\vec{y}|$, $C' = \neg P_1(y_{1,1}, \dots, y_{1,n_1}) \vee \dots \vee \neg P_m(y_{m,1}, \dots, y_{m,n_m})$, the free variables of $P^{\mathcal{S}}$ be \vec{x} (note $|\vec{x}| = n$), and the*

free variables of P_i^S be \vec{x}_i (for $1 \leq i \leq m$). To canonicalize variables, define substitutions:

$$\begin{aligned}\sigma &:= \{y_i \mapsto x_i \mid 1 \leq i \leq n \text{ and there is no } j < i \text{ s.t. } y_j = y_i\} \\ \sigma_i &:= \{x_{i,j} \mapsto y_{i,j} \mid 1 \leq j \leq n_i\} \quad \text{for } 1 \leq i \leq m\end{aligned}$$

The symbolic interpretation that is the result of the operator $\delta(\mathcal{S}, \Lambda \parallel C)$, is defined as follows:

$$\begin{aligned}P^{\delta(\mathcal{S}, \Lambda \parallel C' \vee P(\vec{y}))}(\vec{x}) &:= \left(\pi(\{\vec{y}\}), \bigwedge_{\lambda \in \Lambda} \lambda \wedge \bigwedge_{i=1}^m (P_i^S \sigma_i) \right) \sigma \wedge \Upsilon(\vec{y}, \vec{x}) \\ Q^{\delta(\mathcal{S}, \Lambda \parallel C' \vee P(\vec{y}))}(\vec{z}) &:= \perp \quad \text{for all } Q \neq P \text{ of arity } |\vec{z}|\end{aligned}$$

The goal of the operator $\delta(\mathcal{S}, \Lambda \parallel C)$ is to define an extension of the symbolic interpretation \mathcal{S} such that $\mathcal{S} \cup \delta(\mathcal{S}, \Lambda \parallel C)$ satisfies $\Lambda \parallel C$. Moreover, δ only extends the interpretation over the predicate P of the strictly maximal, positive literal in C and only considers the interpretation \mathcal{S} for predicates Q with $Q \prec P$. To this end, δ defines a linear arithmetic formula $P^{\delta(\mathcal{S}, \Lambda \parallel C' \vee P(\vec{y}))}$ that satisfies two symmetrical properties: On the one hand, it must hold that for all assignments β that satisfy $P^{\delta(\mathcal{S}, \Lambda \parallel C' \vee P(\vec{y}))}$ (i.e. $\models P^{\delta(\mathcal{S}, \Lambda \parallel C' \vee P(\vec{y}))} \beta$) there exists a grounding τ of $\Lambda \parallel C' \vee P(\vec{y})$ where $\mathcal{S} \not\models \Lambda \parallel C' \tau$, $P(\vec{x})\beta = P(\vec{y})\tau$, and \vec{x} are the free variables of P^S . On the other hand, it must hold that for all groundings τ of $\Lambda \parallel C' \vee P(\vec{y})$ with $\mathcal{S} \not\models \Lambda \parallel C' \tau$, there exists an assignment β that satisfies $P^{\delta(\mathcal{S}, \Lambda \parallel C' \vee P(\vec{y}))}$ where $P(\vec{x})\beta = P(\vec{y})\tau$ and \vec{x} are the free variables of P^S .

Note that in the above statements β and τ are generally not the same because the variables \vec{x} used to define P^S are not necessarily the same as the variables appearing in the clause $\Lambda \parallel C$ and literal $P(\vec{y})$. There are three reasons for this that are handled by three different methods in our model construction: First, the variables in \mathcal{S} and $\Lambda \parallel C$ simply do not match, e.g. in $P^S := (x_1 = 0)$ and $\Lambda \parallel C := y_1 > 0 \parallel P(y_1)$. This is handled by the substitution σ in δ that maps all variables in $P(\vec{y})$ to their appropriate variables in P^S , e.g. in the previous example $\sigma := \{y_1 \mapsto x_1\}$ and $P^{\delta(\mathcal{S}, \Lambda \parallel C)} := (y_1 > 0)\sigma$, which is equivalent to $x_1 > 0$. Second, not all variables in $\Lambda \parallel C$ also appear in $P(\vec{y})$, e.g. in $P^S := (x_1 = 0)$ and $\Lambda \parallel C := x_1 = y_1 + 1 \wedge y_1 = 0 \parallel P(x_1)$. This is handled in δ by the projection operator π (Def. 14) that binds all variables that appear in $\Lambda \parallel C$ but not in $P(\vec{y})$, e.g. in the previous example $P^{\delta(\mathcal{S}, \Lambda \parallel C)} := \pi(\{y_1\}, x_1 = y_1 + 1 \wedge y_1 = 0)$, where $\pi(\{y_1\}, x_1 = y_1 + 1 \wedge y_1 = 0) := \exists y_1. (x_1 = y_1 + 1 \wedge y_1 = 0)$, which is equivalent to $x_1 = 1$. Third, some arguments in $P(\vec{y})$ might be the same variable, e.g. in $\Lambda \parallel C := Q(x_1, x_1)$. This is handled in δ by the sharing formula Υ (Def. 15) that expresses which variables in $P^{\delta(\mathcal{S}, \Lambda \parallel C)}$ must be equivalent, e.g. in the previous example $\Upsilon((x_1, x_1), (x_1, x_2)) := (x_1 = x_2)$ and $P^{\delta(\mathcal{S}, \Lambda \parallel C)} := \Upsilon((x_1, x_1), (x_1, x_2))$.

The parts of $P^{\delta(\mathcal{S}, \Lambda \parallel C)}$ that we have not yet discussed are based on the fact that any constrained horn clause $\Lambda \parallel C' \vee P(\vec{y})$ can also be written as an implication of the form $\phi \rightarrow P(\vec{y})$, where $\phi := [\Lambda \wedge P_1(y_{1,1}, \dots, y_{1,n_1}) \wedge \dots \wedge P_m(y_{m,1}, \dots, y_{m,n_m})]$ and $\mathcal{S} \not\models \Lambda \parallel C' \tau$ if and only if $\mathcal{S} \models \phi \tau$. This means the groundings τ of $\Lambda \parallel C'$ not satisfied by \mathcal{S} are also the groundings of ϕ satisfied by \mathcal{S} . It is straightforward to express these groundings with a conjunctive formula based on Λ and the P_i^S . The only challenge is the reverse problem from before,

i.e. mapping the variables of P_i^S to the variables in the literals $P_i(y_{1,1}, \dots, y_{1,n_i})$. This mapping is done in δ by the substitution σ_i .

Now, based on the operator $\delta(\dots)$ for one clause, we can use an inductive definition over the order \prec to define an interpretation \mathcal{S}_N for all clauses. We distinguish the following auxiliary symbolic interpretations: $\mathcal{S}_{\prec P}$ which captures progress up to but excluding the predicate P , Δ_P which captures how P should be interpreted considering $\mathcal{S}_{\prec P}$, and $\mathcal{S}_{\preceq P}$ which captures progress up to and including the predicate P . The symbolic interpretation $\Delta_P^{\Lambda \parallel C}$ is the extension of $\mathcal{S}_{\prec P}$ w.r.t. the single clause $\Lambda \parallel C$.

Definition 17 (Model Construction). *Let N be a finite set of constrained horn clauses. We define symbolic interpretations $\mathcal{S}_{\prec P}$, $\mathcal{S}_{\preceq P}$ and Δ_P for all predicates $P \in \Pi(N)$ by mutual induction over \prec :*

$$\begin{aligned} \mathcal{S}_{\preceq P} &:= \mathcal{S}_{\prec P} \cup \Delta_P & \mathcal{S}_{\prec P} &:= \bigcup_{Q \prec P} \Delta_Q & \Delta_P &:= \bigcup_{\Lambda \parallel C' \vee P(*) \in N} \Delta_P^{\Lambda \parallel C' \vee P(*)} \\ \Delta_P^{\Lambda \parallel C} &:= \begin{cases} \delta(\mathcal{S}_{\prec P}, \Lambda \parallel C) & \text{if } P(\vec{y}) \text{ maximal in } C, \text{ and } \mathcal{S}_{\prec P} \not\models \Lambda \parallel C \\ \mathcal{S}_{\perp} & \text{otherwise} \end{cases} \end{aligned}$$

Finally, based on the above inductive definition of $\mathcal{S}_{\prec P}$ for every predicate symbol $P \in \Pi(N)$, we arrive at an overall interpretation for N .

Definition 18 (Candidate Interpretation). *The candidate interpretation for N (w.r.t \prec), denoted \mathcal{I}_N , is the interpretation associated with the symbolic interpretation $\mathcal{S}_N = \bigcup_{P \in \Pi(N)} \Delta_P$ where P ranges over all predicate symbols occurring in N .*

Note that $\mathcal{S}_N = \mathcal{S}_{\preceq P}$ where P is \prec -maximal in $\Pi(N)$. Obviously, we intend that $\mathcal{S}_N \models N$ if N is saturated (Section 3). Otherwise, i.e. $\mathcal{S}_N \not\models N$, we can use our construction to find a non-redundant inference (Corollary 3). Consider the following two examples, demonstrating how δ sits at the core of the aforementioned inductive definitions of symbolic interpretations.

Example 2 (Dependent Interpretation). *Assume $P \prec Q$ and consider the following set of clauses:*

$$N := \left\{ \begin{array}{ll} 0 \leq x \leq 2, 0 \leq y \leq 2 & \parallel \underline{P(x, y)} \quad (C_1), \\ x_Q \geq x_P + 1, y_Q \geq y_P + 1 & \parallel \underline{P(x_P, y_P)} \rightarrow \underline{Q(x_Q, y_Q)} \quad (C_2) \end{array} \right\}$$

Maximal literals are underlined. Since the maximal literals of C_1 and C_2 are both positive, ordered resolution cannot be applied. The set is saturated. Since P is the \prec -smallest predicate we have $\mathcal{S}_{\prec P} = \mathcal{S}_{\perp}$. Applying the δ operator yields the following interpretation for P :

$$P^{\mathcal{S}_{\preceq P}} = P^{\delta(\mathcal{S}_{\prec P}, C_1)}(x, y) = 0 \leq x \leq 2 \wedge 0 \leq y \leq 2$$

Then, Q is interpreted relative to P . Consider the clause C_2 : For all solutions of its constraint $x_Q \geq x_P + 1, y_Q \geq y_P + 1$ our model must also satisfy its logical part $P(x_P, y_P) \rightarrow Q(x_Q, y_Q)$. The intuition that Q depends on P arises from the implication in the logical part. Whenever the constraint of C_2 and $P(x_P, y_P)$

are satisfied, $Q(x_Q, y_Q)$ must be satisfied. These are exactly the points defined through $\delta(\mathcal{S}_{\prec Q}, C_2)$, based on $\mathcal{S}_{\prec Q} = \mathcal{S}_{\preceq P} = \delta(\mathcal{S}_{\prec P}, C_1)$:

$$\begin{aligned} Q^{\delta(\mathcal{S}_{\prec Q}, C_2)}(x, y) &= \exists x_P, y_P. x \geq x_P + 1 \wedge y \geq y_P + 1 \wedge 0 \leq x_P \leq 2 \wedge 0 \leq y_P \leq 2 \\ &= x \geq 1 \wedge y \geq 1 \end{aligned}$$

Whenever the conjuncts $0 \leq x_P \leq 2$ and $0 \leq y_P \leq 2$ are satisfied, the premise of the implication is true, thus there must be a solution to the interpretation of Q , additionally abiding the constraint of the clause. Since Q is \prec -maximal in N , we arrive at $\mathcal{S}_N = \mathcal{S}_{\preceq Q} = \mathcal{S}_{\preceq P} \cup \delta(\mathcal{S}_{\prec Q}, C_2) = \delta(\mathcal{S}_{\perp}, C_1) \cup \delta(\mathcal{S}_{\preceq P}, C_2)$. See Figure 1a for a visual representation of \mathcal{S}_N .

Example 3 (Unsaturated Clause Set). Assume $P \prec Q$ and consider the following set of clauses:

$$N := \left\{ \begin{array}{ll} x < 0 \parallel \underline{P(x)} & (C_1), \\ x > 0 \parallel \underline{P(x)} & (C_2), \end{array} \quad \begin{array}{ll} x < 1 \parallel \underline{Q(x)} & (C_3), \\ x \leq 0 \parallel \underline{Q(x)} \rightarrow P(x) & (C_4) \end{array} \right\}$$

Maximal literals are underlined. Note that a resolution inference is possible, since the maximal literals of C_3 and C_4 have opposite polarity, use the same predicate symbol, and are trivially unifiable. Thus, in this example we consider the effect of applying our model construction to a clause set that is not saturated. Since P is \prec -minimal, we start with the following steps:

$$\begin{aligned} \mathcal{S}_{\prec P} &= \mathcal{S}_{\perp} \\ P^{\delta(\mathcal{S}_{\prec P}, C_1)}(x) &= x < 0 \\ P^{\delta(\mathcal{S}_{\prec P}, C_2)}(x) &= x > 0 \\ P^{\mathcal{S}_{\preceq P}}(x) &= x < 0 \vee x > 0 \end{aligned}$$

Next, we obtain the following results for Q :

$$\begin{aligned} \mathcal{S}_{\prec Q} &= \mathcal{S}_{\preceq P} \\ Q^{\delta(\mathcal{S}_{\prec Q}, C_1)}(x) &= \perp \\ Q^{\delta(\mathcal{S}_{\prec Q}, C_2)}(x) &= x < 1 \\ Q^{\mathcal{S}_{\preceq Q}}(x) &= \perp \vee x < 1 = x < 1 \end{aligned}$$

See Figure 1b for a visual representation of $\mathcal{S}_N = \mathcal{S}_{\preceq Q}$. Note that $\mathcal{S}_N \not\models C_4$, since we have $\mathcal{S}_N \models Q(0)$ but $\mathcal{S}_N \not\models P(0)$. Thus, by using the constructed model, we can pinpoint clauses that contradict that N is saturated. Adding the resolvent of C_3 and C_4 , i.e. the clause $x \leq 0 \parallel P(x)$ labelled C_5 to N , we instead get

$$P^{\mathcal{S}_{\preceq P}}(x) = x < 0 \vee x > 0 \vee x \leq 0 = \top$$

In the following, we clarify some properties of the construction. We show that all points in $P_N^{\mathcal{I}}$ are necessary and justified in some sense, that \mathcal{I}_N is indeed a model of N , and that \mathcal{I}_N is also the least model of N if N is saturated.

The notion of whether a clause is productive captures whether it contributes something to the symbolic interpretation.

Definition 19 (Productive Clause). Let P be a predicate symbol of arity n . We say that $\Lambda \parallel C$ produces $P(a_1, \dots, a_n)$ if $(a_1, \dots, a_n) \in P^{\Delta_P^{\Lambda \parallel C}}$.

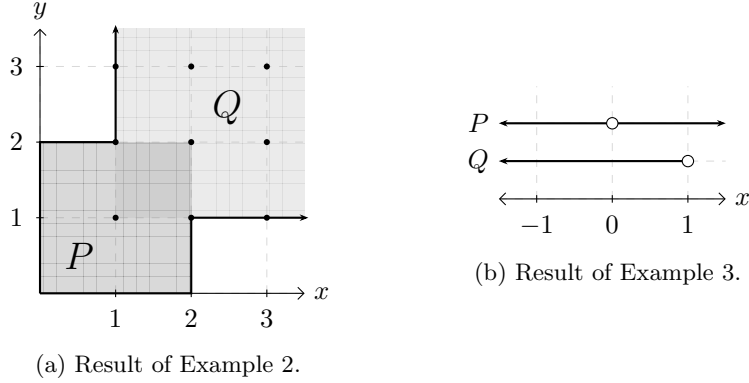


Figure 1: Visual representation of the models resulting from Examples 2 and 3.

Next, we want to formally express that every point in the resulting interpretation is justified in some sense. Firstly, we express that the operator δ will produce points such that every clause is satisfied whenever necessary, i.e. whenever the maximal literal of the clause is $P(*)$ and the maximal literal not satisfied by $\mathcal{S}_{\prec P}$.

Proposition 3. *Let $\Lambda_C \parallel C$ where $C = C' \vee P(\vec{y})$ and $C' \prec P(\vec{y})$. Let τ be a grounding substitution for $\Lambda_C \parallel C$. If $\mathcal{S}_{\prec P} \not\models (\Lambda_C \parallel C)\tau$, then $\models \Lambda_C\tau$ and $\mathcal{S}_{\preceq P} \models P(\vec{y})\tau$, thus $\mathcal{S}_{\preceq P} \models (\Lambda_C \parallel C)\tau$.*

Proof. Let $\Lambda_C \parallel C$ where $C = C' \vee P(\vec{y})$ and $C' \prec P(\vec{y})$. Let τ be a grounding substitution for $\Lambda_C \parallel C$. Assume $\mathcal{S}_{\prec P} \not\models (\Lambda_C \parallel C)\tau$. This implies $\mathcal{S}_{\prec P} \not\models \Lambda_C \parallel C$, thus $\Delta_P^{\Lambda_C \parallel C} = \delta(\mathcal{S}_{\prec P}, \Lambda_C \parallel C)$. Let $\beta_\tau := \{x_i \mapsto y_i\tau \mid 1 \leq i \leq n\}$ and $\phi = \bigwedge_{\lambda \in \Lambda_C} \lambda \wedge \bigwedge_{i=1}^m (P_i^{\mathcal{S}_{\prec P}})\sigma_i$. It remains to show that $\Delta_P \models P(\vec{y})\tau$, i.e. $(\vec{y})\tau \in P^{\delta(\mathcal{S}_{\prec P}, \Lambda_C \parallel C)}$, i.e. $\models ((\pi(\{\vec{y}\}), \phi))\sigma \wedge \Upsilon(\vec{y}, \vec{x})\beta_\tau$. We proceed in two steps, one per conjunct:

1. To see $\models (\Upsilon(\vec{y}, \vec{x}))\beta_\tau$: Let $1 \leq i < j \leq n = |\vec{y}| = |\vec{x}|$. By definition of Υ , for each conjunct $x_i = x_j$ in $\Upsilon(\vec{y}, \vec{x})$, there are two variables $y_i = y_j$ from \vec{y} . The conjunct $x_i = x_j$ is satisfied by β_τ , since β_τ and τ are functions, therefore $(x_i)\beta_\tau = (x_j)\beta_\tau$.
2. To see $\models (\pi(\{\vec{y}\}), \phi)\sigma\beta_\tau$: From $\mathcal{S}_{\prec P} \not\models (\Lambda_C \parallel C)\tau$, we know $\models \Lambda_C\tau$, thus $\models \lambda\tau$ for all $\lambda \in \Lambda_C$, and that $\mathcal{S}_{\prec P} \not\models (L_i)\tau$ for all literals L_i from C' . Given that $P_i^{\mathcal{S}_{\prec P}}$ encodes $\mathcal{S}_{\prec P} \not\models L_i$ it follows that $\models P_i^{\mathcal{S}_{\prec P}}\sigma_i\tau$ for all $1 \leq i \leq m$.

Thus, we may construct a potential witness for all existential quantifiers introduced by the application of π based on τ : $\gamma_\tau := \{z_i \mapsto z_i\tau \mid z_i \in \text{vars}(\Lambda_C \parallel C) \setminus \{\vec{y}\}\}$.

By definition of σ and β_τ , we have $(y_i)\sigma\beta_\tau = (y_i)\tau$ for all y_i in \vec{y} . Thus $(\phi)\gamma_\tau\sigma\beta_\tau$ is equivalent to $(\phi)\tau$. Again, reasoning from $\mathcal{S}_{\prec P} \not\models (\Lambda_C \parallel C)\tau$ we arrive at $\models (\phi)\tau$, thus $\models (\pi(\{\vec{y}\}), \phi)\sigma\beta_\tau$ with witness γ_τ .

Hence $\models ((\pi(\{\vec{y}\}), \phi))\sigma \wedge \Upsilon(\vec{y}, \vec{x})\beta_\tau$, thus $\Delta_P^{\Lambda_C \parallel C} \models \Lambda_C \parallel C$, and by definition of $\mathcal{S}_{\preceq P}$, also $\mathcal{S}_{\preceq P} \models \Lambda_C \parallel C$. \square

Secondly, we express that for every point in P_N^I , it is justified in the sense that there is a clause that produced the point, i.e. this clause would otherwise not be satisfied by the resulting interpretation.

Proposition 4. *If $\mathcal{S}_{\leq P} \models P(\vec{a})$, then there exists a clause $\Lambda_C \parallel C$ where $C = C' \vee P(\vec{y})$ and $C' \prec P(\vec{y})$, and there exists a grounding τ for $\Lambda_C \parallel C$, such that $P(\vec{a}) = P(\vec{y})\tau$ and $\mathcal{S}_{\prec P} \not\models (\Lambda_C \parallel C)\tau$.*

Proof. Assume $\mathcal{S}_{\leq P} \models P(\vec{a})$.

Firstly, we argue that $\vec{a} \in P^{\Delta P}$: For the construction of $\mathcal{S}_{\prec P}$ only predicates Q s.t. $Q \prec P$ are considered, thus $P^{\mathcal{S}_{\prec P}} = \emptyset$. By definition of $\mathcal{S}_{\leq P}$ as $\mathcal{S}_{\leq P}$ as the union of $\mathcal{S}_{\prec P}$ and Δ_P and thus $P^{\mathcal{S}_{\leq P}} = P^{\mathcal{S}_{\prec P}} \cup P^{\Delta_P}$. Together with $P^{\mathcal{S}_{\leq P}} = \emptyset$ we have $P^{\mathcal{S}_{\leq P}} = P^{\Delta_P}$, hence $\vec{a} \in P^{\Delta_P}$.

Secondly, we argue for the existence of $\Lambda_C \parallel C$: Δ_P is defined as the union of $\Delta_P^{\Lambda \parallel D' \vee P(*)}$ where $\Lambda \parallel D' \vee P(*) \in N$. Thus there must be at least one $\delta(\mathcal{S}_{\prec P}, \Lambda \parallel D' \vee L)$ with $L = P(\vec{y})$, $D' \prec L$, $\mathcal{S}_{\prec P} \not\models \Lambda \parallel D' \vee L$, s.t. $\vec{a} \in P^{\delta(\mathcal{S}_{\prec P}, \Lambda \parallel D' \vee L)}$. We identify $\Lambda \parallel D' \vee L$ with $\Lambda_C \parallel C$.

Thirdly, we construct τ , and show $\mathcal{S}_{\prec P} \not\models (\Lambda_C \parallel C)\tau$ by unfolding the interpretation of P by $\mathcal{S}_{\prec P}$: Let $\beta_x = \{x_1 \mapsto a_1, \dots, x_n \mapsto a_n\}$. From $\vec{a} \in \Delta_P^{\Lambda_C \parallel C}$ we know that β_x is a solution for $(\pi(\{\vec{y}\}, \bigwedge_{\lambda \in \Lambda_C} \lambda \wedge \bigwedge_{i=1}^m (P_i^{\mathcal{S}_{\prec P}})\sigma_i))\sigma \wedge \Upsilon(\vec{y}, \vec{x})$. Thus, $\sigma\beta_x$ is a solution for $\pi(\{\vec{y}\}, \bigwedge_{\lambda \in \Lambda_C} \lambda \wedge \bigwedge_{i=1}^m (P_i^{\mathcal{S}_{\prec P}})\sigma_i)$. Let

$$\gamma := \{z \mapsto c_z \mid z \in \text{vars}(C) \setminus \text{vars}(L)\}$$

be a witness for the existential quantifiers introduced by π , i.e. $\models (\bigwedge_{\lambda \in \Lambda_C} \lambda \wedge \bigwedge_{i=1}^m (P_i^{\mathcal{S}})\sigma_i)\gamma\sigma\beta_x$. The composition $\tau := \gamma\sigma\beta_x$ then maps all variables from C and is a solution for $\bigwedge_{\lambda \in \Lambda_C} \lambda \wedge \bigwedge_{i=1}^m (P_i^{\mathcal{S}})\sigma_i$. Consequently, τ is a solution for all conjuncts:

- $\models \lambda\tau$ for all $\lambda \in \Lambda_C$ immediately gives $\models \Lambda_C\tau$.
- $\models (P_i^{\mathcal{S}}\sigma_i)\tau$ for all $1 \leq i \leq m$ witnesses $\mathcal{S}_{\prec P} \not\models L_i\tau$, because the polarity of $P_i^{\mathcal{S}_{\prec P}}$ is opposite of the polarity of L_i by definition of δ .

We have $P^{\mathcal{S}_{\prec P}} = \emptyset$, and that L is positive, thus $\mathcal{S}_{\prec P} \not\models L$. Together with the two above facts we arrive at $\mathcal{S}_{\prec P} \not\models (\Lambda_C \parallel C)\tau$.

To see that $P(\vec{y})\tau = P(\vec{a})$, consider that $(y_i)\sigma = x_i$, $(x_i)\beta_x = a_i$, and γ has no effect on \vec{y} by definition. Thus $(y_i)\sigma\beta_x = (y_i)\tau = a_i$. In case there are two (or more, analogously) variables y_i, y_j in \vec{y} where $y_i = y_j$ and $i < j$, $a_i = a_j$ is guaranteed: $(y_i)\tau = (x)\beta_x = a_i$ directly by definition of σ and β_x . y_j is not in the domain of σ , however the equalities generated by $\Upsilon(\vec{y}, \vec{x})$ ensure that $(x_i)\beta_x = (x_j)\beta_x$. \square

The following proposition relates clauses to the interpretation constructed up to their respective maximal predicate. Together with the fact that the interpretation of P stays fixed when considering $Q \prec P$.

Proposition 5. *For every clause $\Lambda_C \parallel C \in N$ with maximal predicate P , if $\mathcal{S}_{\leq P} \models \Lambda_C \parallel C$, then $\mathcal{I}_N \models N$.*

Also, observe that once the maximal predicate P of a given clause is interpreted by $\mathcal{S}_{\leq P}$, the interpretation of the clause does not change for $\mathcal{S}_{\leq Q}$ where $Q \succ P$.

Corollary 1. *Let $P \prec Q \preceq R$, and P be maximal in clause C . If $\mathcal{S}_{\preceq P} \models \Lambda_C \parallel C$ or $\mathcal{S}_{\prec Q} \models \Lambda_C \parallel C$, then $\mathcal{S}_{\prec R} \models \Lambda_C \parallel C$ and $\mathcal{S}_{\preceq R} \models \Lambda_C \parallel C$.*

Proposition 6. *Let $P \prec Q \preceq R$. $P^{\mathcal{S}_{\preceq P}} = P^{\mathcal{S}_{\prec Q}} = P^{\mathcal{S}_{\preceq Q}} = P^{\mathcal{S}_{\prec R}} = P^{\mathcal{S}_{\preceq R}}$.*

Proof. Generally, for a predicate symbol S , we have $P_S^\Delta = \emptyset$ unless $S = P$ by definition of Δ_S .

$\mathcal{S}_{\prec Q}$ is defined as the union of all Δ_S for $S \prec Q$, thus $P^{\Delta_P} = P^{\mathcal{S}_{\preceq P}} = P^{\mathcal{S}_{\prec Q}}$. $\mathcal{S}_{\preceq Q}$ is defined as the union of $\mathcal{S}_{\prec Q}$ and Δ_Q .

By definition of Δ_Q , the only predicate that Δ_Q may interpret as non-empty is Q . In particular, i.e. since $P \prec Q$, we have $P^{\Delta_Q} = \emptyset$ thus $P^{\mathcal{S}_{\prec Q}} = P^{\mathcal{S}_{\preceq Q}}$. To see that $P^{\mathcal{S}_{\preceq Q}} = P^{\mathcal{S}_{\prec R}} = P^{\mathcal{S}_{\preceq R}}$, the same reasoning applies, i.e. $P^{\Delta_R} = \emptyset$ unless $P = R$. \square

Corollary 2. *Let $P \prec Q \preceq R$. If $\mathcal{S}_{\preceq P} \models [\neg]P(\vec{x})\sigma$ or $\mathcal{S}_{\prec Q} \models [\neg]P(\vec{x})\sigma$, then $\mathcal{I}_{\prec R} \models [\neg]P(\vec{x})\sigma$ and $\mathcal{I}_{\preceq R} \models [\neg]P(\vec{x})\sigma$.*

Proof. Corollary of Proposition 6. \square

Proposition 7. *Let $\Lambda_D \parallel D$ be a horn clause with $D = D' \vee P(\vec{y})$. If $\Lambda_D \parallel D$ produces $P(\vec{y})\tau$, then for all grounding substitutions σ , such that $\mathcal{S}_{\prec P} \not\models (\Lambda_D \parallel D')\sigma$, for all Q such that $P \preceq Q$, we have $\mathcal{S}_{\preceq Q} \not\models (\Lambda_D \parallel D')\sigma$.*

Proof. Since $\Lambda_D \parallel D$ produces $P(\vec{y})\tau$, all literals in D' are strictly \prec -smaller than $P(\vec{y})$, and $\mathcal{S}_{\prec P} \not\models \Lambda_D \parallel D$.

Let σ be a grounding substitution such that $\mathcal{S}_{\prec P} \not\models (\Lambda_D \parallel D')\sigma$.

Assume, towards a contradiction, that the proposition does not hold, i.e. $P \preceq Q$ and $\mathcal{S}_{\preceq Q} \models (\Lambda_D \parallel D')\sigma$. Then we have $\models \Lambda_D \sigma$ and $\mathcal{S}_{\preceq Q} \models ([\neg]R(*))\sigma$ for some literal $[\neg]R(*)$ in D' .

However, by assumption, all literals in D' are strictly \prec -smaller than $P(\vec{y})$, i.e. $R \prec P$. Thus their respective interpretation is contained in $\mathcal{S}_{\prec P}$ by construction, i.e. $R^{\mathcal{S}_{\preceq R}} = R^{\mathcal{S}_{\prec P}} = R^{\mathcal{S}_{\preceq Q}}$. This contradicts $\mathcal{S}_{\prec P} \not\models (\Lambda_D \parallel D')\sigma$. \square

Proposition 8. *Let \prec be a clause ordering and N be a finite set of constrained horn clauses. If (1.) N is saturated w.r.t. \prec -resolution, and (2.) there is no $\Lambda \parallel \perp \in N$ where Λ is satisfiable, then for every clause $\Lambda_C \parallel C \in N$, $\models \Lambda_C \parallel C$ or P is maximal in C and $\mathcal{S}_{\preceq P} \models \Lambda_C \parallel C$.*

Proof. Assume Premises 1 and 2. Let $\Lambda_C \parallel C \in N$. We distinguish two cases:

1. $C = \perp$. By Premise 2, Λ_C is unsatisfiable, thus $\models \Lambda_C \parallel C$.
2. $C = C' \vee L$ where L is maximal in C , and P is the predicate symbol associated with L , i.e. P is maximal in C . In case Λ_C is unsatisfiable, $\models \Lambda_C \parallel C$. In the following, we thus consider Λ_C satisfiable. We distinguish two cases:
 - 2.1. $L = P(\vec{y})$. We distinguish two cases:
 - 2.1.1. $\mathcal{S}_{\prec P} \models \Lambda_C \parallel C$. Corollary 2 applies.
 - 2.1.2. $\mathcal{S}_{\prec P} \not\models \Lambda_C \parallel C$. Proposition 3 applies.

2.2. $L = \neg P(\vec{z})$. Assume, towards a contradiction, that the proposition does not hold, i.e. $\mathcal{S}_{\leq P} \not\models \Lambda_C \parallel C$. Let σ be a grounding substitution such that $\mathcal{S}_{\leq P} \not\models (\Lambda_C \parallel C)\sigma$. It follows that $\mathcal{S}_{\leq P} \not\models (C')\sigma$ and $\mathcal{S}_{\leq P} \not\models L\sigma$, thus $\mathcal{S}_{\leq P} \models (P(\vec{z}))\sigma$. Assume, without loss of generality:

- *minimality of P* : There is no predicate symbol $Q \prec P$ such that there is a clause $\Lambda_D \parallel D \in N$ with maximal predicate Q and $\mathcal{S}_{\leq Q} \not\models \Lambda_D \parallel D$, i.e. for all $Q \prec P$, $\mathcal{S}_{\leq Q} \models \{\Lambda_D \parallel D \in N \mid Q \text{ maximal in } D\}$,
- *minimality of $(\Lambda_C \parallel C)\sigma$* : There is no clause $\Lambda_D \parallel D \in N$ with maximal predicate P and grounding σ_D , s.t. $(\Lambda_D \parallel D)\sigma_D \prec (\Lambda_C \parallel C)\sigma$ and $\mathcal{S}_{\leq P} \not\models (\Lambda_D \parallel D)\sigma_D$, i.e. $\mathcal{S}_{\leq P} \models \text{gnd}(\{\Lambda_D \parallel D \in N \mid P \text{ maximal in } C\})^{\prec(\Lambda_C \parallel C)\sigma}$,
- *maximality of $L\sigma$* : there is no literal L' in $C'\sigma$ s.t. $\neg P(\vec{z})\sigma \prec L'$.

By Proposition 4 there is a clause $\Lambda_D \parallel D \in N$ where $D = D' \vee P(\vec{y})$, $D' \prec P(\vec{y})$, and Λ_D is satisfiable, that produces $P(\vec{z})\sigma$. Assume, without loss of generality, that the sets of variables occurring in $\Lambda_C \parallel C$ and $\Lambda_D \parallel D$ are disjoint, i.e. $\text{vars}(\Lambda_C \parallel C) \cap \text{vars}(\Lambda_D \parallel D) = \emptyset$. Let τ be the substitution that maps $P(\vec{y})$ to $P(\vec{z})\sigma$. Since $P(\vec{y})\tau = P(\vec{z})\sigma$, there exists a most general unifier between $P(\vec{y})$ and $P(\vec{z})$, which we call σ' . Then, since σ' is most general, there is a substitution τ' such that the substitution $\sigma'\tau'$ is equivalent to σ when restricted to $\text{vars}(\Lambda_C \parallel C)$ and equivalent to τ when restricted to $\text{vars}(\Lambda_D \parallel D)$. We have $P(\vec{y})\sigma'\tau' = P(\vec{y})\tau = P(\vec{z})\sigma$. Consider the following \prec -resolution inference:

$$\frac{\Lambda_C \parallel C' \vee \neg P(\vec{z}) \quad \Lambda_D \parallel D' \vee P(\vec{y}) \quad \sigma' = \text{mgu}(\neg P(\vec{z}), \text{comp}(P(\vec{y})))}{(\Lambda_C, \Lambda_D \parallel C' \vee D')\sigma'}$$

Let $(\Lambda_C, \Lambda_D \parallel C' \vee D')\sigma' = \Lambda_R \parallel R$. Since $\models \Lambda_C\sigma$ and $\models \Lambda_D\tau$ (by Proposition 4), we have $\models \Lambda_R\tau'$. Since $\Lambda_D \parallel D$ produces P , we have $\mathcal{S}_{\leq P} \not\models (\Lambda_D \parallel D')\tau$. By Proposition 7, we have that, $\mathcal{S}_{\leq P} \not\models (\Lambda_D \parallel D')\tau$ and $\mathcal{S}_{\leq P} \not\models (\Lambda_D \parallel D')\sigma'\tau'$. Thus $\mathcal{S}_{\leq P} \not\models \Lambda_R \parallel R\tau'$ and $\mathcal{S}_{\leq P} \not\models \Lambda_R \parallel R$.

We distinguish two cases:

2.2.1. $R = \perp$. We distinguish two cases:

2.2.1.1. $\Lambda_R \parallel \perp \in N$. Contradicts Premise 2.

2.2.1.2. $\Lambda_R \parallel \perp \notin N$. Then, by Premise 1, we have that $\Lambda_R \parallel \perp$ is redundant, i.e. $N^{\prec \Lambda_R \parallel \perp} \models \Lambda_R \parallel \perp$. However, since there is no clause that is \prec -smaller than $\Lambda_R \parallel \perp$, we have $N^{\prec \Lambda_R \parallel \perp} = \emptyset$, which contradicts Premise 1.

2.2.2. $R \neq \perp$. $\neg P(\vec{z})\sigma$ occurs less often in $R\tau'$ than it occurs in $C\sigma$. The reason being that the number of occurrences in $C'\sigma'$ is one less than the number of occurrences in $C\sigma$, while there are no occurrences in $D'\sigma'$ since we know that $P(\vec{y})\sigma' \prec D\sigma'$, and $P(\vec{y})\sigma' \prec \neg P(\vec{z})\sigma$. By maximality of $\neg P(\vec{z})\sigma$, also $\neg P(\vec{y})\tau' \preceq R\tau'$. Therefore we have $\Lambda_R \parallel R \prec \Lambda_C \parallel C$. By minimality of $(\Lambda_C \parallel C)\sigma$, Case 2.1., and Corollary 1, we have $\mathcal{S}_{\leq P} \models \text{gnd}(N)^{\prec(\Lambda_C \parallel C)\sigma}$.

We distinguish two cases:

2.2.2.1. $\Lambda_R \parallel R \in N$. Contradicts minimality of $(\Lambda_C \parallel C)\sigma$. Contradicts $\mathcal{S}_{\leq P} \models \text{gnd}(N)^{\prec(\Lambda_C \parallel C)\sigma}$ since $(\Lambda_R \parallel R)\tau' \prec (\Lambda_C \parallel C)\sigma$.

2.2.2.2. $\Lambda_R \parallel R \notin N$. By assumption of Premise 1, $\Lambda_R \parallel R$ is redundant w.r.t. N and \prec , i.e. $N^{\prec \Lambda_R \parallel R} \models \Lambda_R \parallel R$. $\mathcal{S}_{\leq P} \models \text{gnd}(N)^{\prec(\Lambda_C \parallel C)\sigma}$ contradicts $\text{gnd}(N)^{\prec(\Lambda_R \parallel R)\tau'} \models (\Lambda_R \parallel R)\tau'$ and $N^{\prec \Lambda_R \parallel R} \models \Lambda_R \parallel R$. Therefore $\Lambda_R \parallel R$ is not redundant, which contradicts Premise 1.

□

With the above propositions we show that indeed $\mathcal{I}_N \models N$.

Let \prec be a clause ordering and N be a set of constrained horn clauses. If (1.) N is saturated w.r.t. \prec -resolution, and (2.) $\square \notin N$, then $\mathcal{I}_N \models N$.

Proof. By Proposition 8 and Proposition 5. □

For clauses with positive maximal literal, the fact that they are satisfied by \mathcal{I}_N follows from Proposition 3. For clauses with maximal literal $\neg P(*)$, we prove this theorem by contradiction: If there is a minimal clause $\Lambda_C \parallel C$ such that $\mathcal{S}_N \not\models \Lambda_C \parallel C$. We can then exploit Proposition 4 to find the smallest clause $\Lambda_D \parallel D$ that produced the respective instance $P(\vec{a})$. Applying hierarchic \prec -resolution to $\Lambda_C \parallel C$ and $\Lambda_D \parallel D$ then yields a non-redundant clause. This idea then leads to the following theorem.

Corollary 3. *Let \prec be a clause ordering and N be a set of constrained horn clauses. If (1.) $\mathcal{I}_N \not\models N$, and (2.) $\square \notin N$, then there exist two clauses $\Lambda_C \parallel C$, $\Lambda_D \parallel D \in N$ such that: (1.) $\Lambda_C \parallel C$ is the smallest clause not satisfied by \mathcal{I}_N , i.e. there exists a grounding τ such that $\mathcal{I}_N \not\models (\Lambda_C \parallel C)\tau$, but there does not exist a clause $\Lambda_{C'} \parallel C' \in N$ with grounding τ' , such that $\mathcal{I}_N \not\models (\Lambda_{C'} \parallel C')\tau'$ and $(\Lambda_{C'} \parallel C')\tau' \prec (\Lambda_C \parallel C)\tau$, (2.) $\neg P(\vec{a})$ is the maximal literal of $(\Lambda_C \parallel C)\tau$, (3.) $\Lambda_D \parallel D$ is the minimal clause that produces $P(\vec{a})$, (4.) \prec -resolution is applicable to $\Lambda_C \parallel C$ and $\Lambda_D \parallel D$, and (5.) the resolvent of $\Lambda_C \parallel C$ and $\Lambda_D \parallel D$ is not redundant w.r.t. N .*

Proof. Similar to Case 2.2. in the proof of Proposition 8. □

Additionally, we show that \mathcal{I}_N is the least model of N , establishing a connection between our approach and the literature on constrained horn clauses (see [28, Section 4] and [13, Section 2.4.1]) and logic programming (see [35, § 6, p. 37]).

\mathcal{I}_N is the least model of N .

Proof. Assume, towards a contradiction, that \mathcal{I}_N is not the least model of N , i.e. there exists an interpretation \mathcal{I} such that $\mathcal{I} \subset \mathcal{I}_N$ and $\mathcal{I} \models N$. Since $\mathcal{I} \subset \mathcal{I}_N$, there is a predicate symbol P and a point \vec{a} such that $\vec{a} \in P^{\mathcal{I}_N}$, i.e. $\mathcal{I}_N \models P(\vec{a})$ but $\vec{a} \notin P^{\mathcal{I}}$, i.e. $\mathcal{I} \not\models P(\vec{a})$. Assume, w.l.o.g., that P is minimal, i.e. $Q^{\mathcal{I}_N} = Q^{\mathcal{I}}$ for all $Q \prec P$. By Proposition 4 from $\mathcal{I}_N \models P(\vec{a})$ it follows that there is a clause $\Lambda_C \parallel C \in N$ such that $C = C' \vee P(\vec{y})$, $C' \prec P(\vec{y})$, τ is a grounding for $\Lambda_C \parallel C$, $P(\vec{a}) = P(\vec{y})\tau$, and $\mathcal{S}_{\prec P} \not\models (\Lambda_C \parallel C)\tau$. From $\mathcal{S}_{\prec P} \not\models (\Lambda_C \parallel C)\tau$ we know that $\mathcal{S}_{\prec P} \not\models (\Lambda_C \parallel C')\tau$ and by Proposition 6 we have $\mathcal{I}_N \not\models (\Lambda_C \parallel C')\tau$. Since $C' \prec P(\vec{y})$ and by minimality of P , we know that \mathcal{I}_N and \mathcal{I} agree on $\Lambda_C \parallel C'$, i.e. $\mathcal{I} \not\models (\Lambda_C \parallel C')\tau$. However, $\mathcal{I} \models N$, which implies $\mathcal{I} \models (\Lambda_C \parallel C)\tau$, requires $\mathcal{I} \models P(\vec{a})$ which contradicts the assumption $\mathcal{I} \not\models P(\vec{a})$. □

Fermüller and Leitsch define four postulates (see [18], [11, Section 5.1, p. 234]) regarding automated model building. Within their taxonomy, our approach falls into the category of *constrained atomic representations*, thus the first postulate *uniqueness* is trivial [11, p. 235]. The second postulate, *atom test* is fulfilled only insofar as constraint solving is considered fast. The third and fourth postulate, *formula evaluation* and *equivalence test* are again tied to constraint solving and fulfilled.

4 Conclusion

We have presented the first model construction approach to Horn clauses with linear arithmetic constraints based on hierarchic superposition, Definition 18. The linear arithmetic constraints may range over the reals, rationals, or integers. The computed model is the canonical least model of the saturated Horn clause set, Section 3. Clauses can be effectively evaluated with respect to the model, Proposition 2. This offers a way to explore the properties of a saturated clause set, e.g., if the set represents a failed refutation attempt.

Future Work It is straightforward to see that any symbolic LQA model is also a symbolic LRA model. (This holds due to convexity of conjunctions of ground LQA atoms.) So even if the axiom of choice is not assumed, there is an alternative way to obtain a satisfying interpretation for a HBS(LRA) clause set: we simply treat it as an HBS(LQA) clause set, saturate it and construct its model based on HBS(LQA).

In this work, we restrict ourselves to only one sort LA per set of clauses. An extension to a many-sorted setup, e.g. including first-order variables with sort \mathcal{F} is possible. This can even be simulated, by encoding first-order constants as concrete natural numbers via a bijection to \mathbb{N} , since $\mathbb{N} \subset \mathcal{U}$. By not placing any arithmetic constraints on the variables used for the encoding, it can be read off and mapped back from the resulting model.

One obvious challenge is relaxation of the restriction to Horn clauses. With respect to superposition saturation there is typically no difference in the sense that if a Horn fragment can always be finitely saturated, so can the non-Horn fragment be. However, our proposed ordering for the model construction at the granularity of predicate symbols will not suffice in this general case, and the key to overcome this challenge seems to be the appropriate treatment of clauses with maximal literals of the same predicate. Backtracking on the selection of literals might also be sufficient.

The approach we presented does not exploit features of linear arithmetic beyond equality and the existence of a well-founded order for the underlying universe \mathcal{U} . The results may therefore be adapted to other constraint domains such as non-linear arithmetic.

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